LECTURES ON REAL OPTIONS: PART II — TECHNICAL ANALYSIS

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Use of Option Pricing Methods

- Hard to handle options with NPV methods. Need option pricing methods.
- We will illustrate this with a simple case:
  The Option to Invest.

  Example: You are deciding whether to build a plant that would produce widgets. The plant can be built quickly, and will cost \$1 million. A careful analysis shows that the present value of the cash flows from the plant, if it were up and running today, is \$1.2 million. Should you build the plant?

  Answer: Not clear.

You Have an Option to Invest

- Issue is whether you should exercise this option.
- If you exercise the option, it will cost you $I = \$1$ million. You will receive an asset whose value today is $V = \$1.2$ million. Of course V might go up or down in the future, as market conditions change.
- Compare to call option on a stock, where P is price of stock and EX is exercise price:

Option	Payoff
Call option on stock:	Max (P-EX, 0)
Option to invest in factory	Max (V-I, 0)

Nature of Option to Invest

- Unlike call option on a stock, option to invest may be long-lived, even perpetual.
- Why does the firm have this option?
  - Patents, and technological know-how.
  - A license or copyright.
  - Land or mineral rights.
  - Firm's market position, reputation.
  - Scale economies.
  - Managerial know-how.
- In general, a firm's options to invest can account for a large part of the firm's market value.

Project Value and Investment Decision

- To solve this problem, must model the value of the project and its evolution over time.
- Given the dynamics of the project's value, we can value the option to invest in the project.
- Valuing the option to invest requires that we find the optimal investment rule, i.e., the rule for when to invest.
  - "When" does not mean determining the point in time that investment should occur.
  - It means finding the critical value of the project that should trigger investment.

Dynamics of Project Value, V

- Value of project, V, will evolve over time.
  - μ = Expected return on V. This expected return will be consistent with the project's (nondiversifiable) risk.
  - δ = Payout rate on project. This is the rate of cash payout, as fraction of V.
  - So $\mu = \delta +$ expected rate of capital gain.
- First, suppose there is No Risk.
  - Then rate of capital gain is:

$$\frac{\Delta V}{V} = (\mu - \delta)\Delta t$$

and $\mu = r_f$, the risk-free interest rate.

Dynamics of Project Value, V (Continued)

- Now, suppose V is risky. Then:

$$\frac{\Delta V}{V} = (\mu - \delta)\Delta t + \sigma e_t$$

where e_t is random, zero-mean. So V follows a random walk, like the price of a stock.

  - If all risk is diversifiable, $\mu = r_f$.
  - If there is nondiversifiable risk, $\mu > r_f$.
  - Formally, write process for V as:

$$\frac{dV}{V} = (\mu - \delta)dt + \sigma dz$$

  where $dz = e_t\sqrt{dt}$ is the increment of a Wiener process, and e_t normally distributed, with $(dz)^2 = dt$.

  - So V follows a geometric Brownian motion (GBM).
  - Note $(dV)^2 = \sigma^2 V^2 dt$.

Mathematical Background

- **Wiener Process**: If $z(t)$ is a Wiener process, any change in z, Δz, over time interval Δt, satisfies
  1. The relationship between Δz and Δt is given by:

     $$\Delta z = \epsilon_t \sqrt{\Delta t},$$

     where ϵ_t is a normally distributed random variable with zero mean and a standard deviation of 1.
  2. The random variable ϵ_t is serially uncorrelated, i.e., $\mathcal{E}[\epsilon_t \epsilon_s] = 0$ for $t \neq s$. Thus values of Δz for any two different time intervals are independent.

Mathematical Background (continued)

- What do these conditions imply for change in z over an interval of time T?
- Break interval up into n units of length Δt each, with $n = T/\Delta t$. Then the change in z over this interval is

$$z(s+T) - z(s) = \sum_{i=1}^{n} \epsilon_i \sqrt{\Delta t} \tag{1}$$

The ϵ_i's are independent of each other. By the Central Limit Theorem, the change $z(s+T) - z(s)$ is normally distributed with mean 0 and variance $n\,\Delta t = T$.

  - This result, which follows from the fact that Δz depends on $\sqrt{\Delta t}$ and not on Δt, is particularly important; *the variance of the change in a Wiener process grows linearly with the time horizon.*

Mathematical Background (continued)

- By letting Δt become infinitesimally small, we can represent the increment of a Wiener process, dz, as:

$$dz = \epsilon_t \sqrt{dt} \tag{2}$$

  - Since ϵ_t has zero mean and unit standard deviation, $\mathcal{E}(dz) = 0$, and $\mathcal{V}[dz] = \mathcal{E}[(dz)^2] = dt$.
  - Note that a Wiener process has no time derivative in a conventional sense; $\Delta z / \Delta t = \epsilon_t \, (\Delta t)^{-1/2}$, which becomes infinite as Δt approaches zero.

- <u>Generalization: Brownian Motion with Drift</u>.

$$dx = \alpha \, dt + \sigma \, dz, \tag{3}$$

where dz is the increment of a Wiener process, α is the drift parameter, and σ the variance parameter.

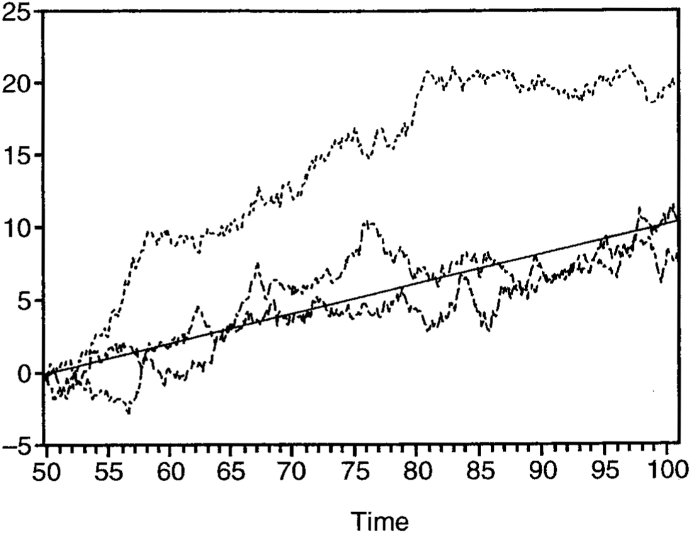
Mathematical Background (continued)

- Over any time interval Δt, Δx is normally distributed, and has expected value $\mathcal{E}(\Delta x) = \alpha\,\Delta t$ and variance $\mathcal{V}(\Delta x) = \sigma^2 \Delta t$.
- Figure 1 shows three sample paths of eq. (3), with $\alpha = 0.2$ per year, and $\sigma = 1.0$ per year. Each sample path was generated by taking a Δt of one month, and then calculating a trajectory for $x(t)$ using

$$x_t = x_{t-1} + .01667 + .2887\,\epsilon_t, \tag{4}$$

with $x_{1950} = 0$. In eq. (4), at each time t, ϵ_t is drawn from a normal distribution with zero mean and unit standard deviation. (Note: α and σ are in monthly terms. Trend of .2 per year implies 0.0167 per month; S.D. of 1.0 per year implies variance of 1.0 per year, hence variance of $\frac{1}{12} = .0833$ per month, so monthly S.D. is $\sqrt{0.0833} = 0.2887$.) Also shown is a trend line, i.e., eq. (4) with $\epsilon_t = 0$.

Sample Path of Brownian Motion with Drift



Mathematical Background (continued)

- Fig. 2 shows a forecast of this process. A sample path was generated from 1950 to the end of 1974, again using eq. (4), and then forecasts of $x(t)$ were constructed for 1975 to 2000. The forecast of x for T months beyond Dec. 1974 is

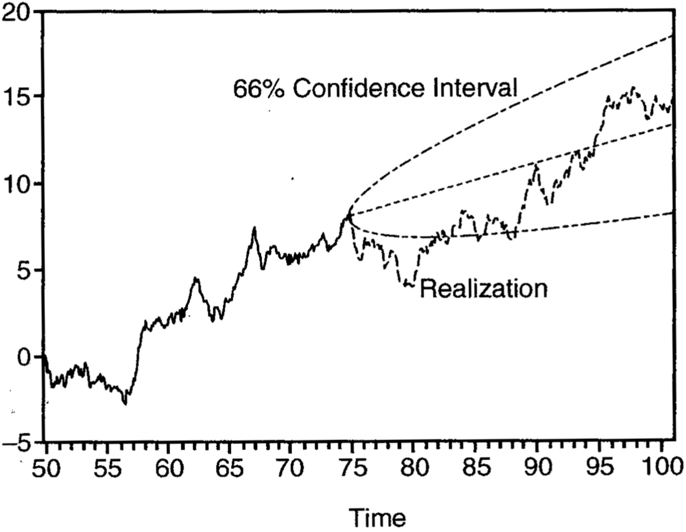
$$\hat{x}_{1974+T} = x_{1974} + .01667\,T.$$

- The graph also shows a 66 percent forecast confidence interval, i.e., the forecasted $x(t)$ plus or minus one standard deviation. Recall that variance grows linearly with the time horizon, so the standard deviation grows as the *square root* of the time horizon. Hence the 66 percent confidence interval is

$$x_{1974} + .01667\,T \pm .2887\,\sqrt{T}.$$

  One can similarly construct 90 or 95 percent confidence intervals.

Forecast of Brownian Motion with Drift



Mathematical Background (continued)

- Generalized Brownian Motion: Ito Process.

$$dx = a(x,t)dt + b(x,t)dz$$

  - Mean of dx: $\mathcal{E}(dz) = 0$ so $\mathcal{E}(dx) = \alpha(x,t)dt$.
  - Variance of dx: $\mathcal{V}(dx) = \mathcal{E}[dx^2] - (\mathcal{E}[dx]^2)$, which hass terms in dt, $(dt)^2$, and $(dt)(dz)$, which is of order $(dt)^{3/2}$. For dt infinitesimally small, terms in $(dt)^2$ and $(dt)^{3/2}$ can be ignored, so

$$\mathcal{V}[dx] = b^2(x,t)\,dt.$$

- Geometric Brownian Motion (GBM).
  $a(x,t) = \alpha x$, and $b(x,t) = \sigma x$, so

$$dx = \alpha\,x\,dt + \sigma\,x\,dz. \tag{5}$$

  - Percentage changes in x, $\Delta x/x$, are normally distributed, so absolute changes in x, Δx, are *lognormally* distributed.

Mathematical Background (continued)

- If $x(0) = x_0$, the expected value of $x(t)$ is

$$\mathcal{E}[x(t)] = x_0\, e^{\alpha t},$$

and the variance of $x(t)$ is

$$\mathcal{V}[x(t)] = x_0^2\; e^{2\alpha t}\; (e^{\sigma^2 t} - 1).$$

We can calculate the expected present discounted value of $x(t)$ over some period of time. For example,

$$\mathcal{E}\left[\int_0^\infty x(t)\, e^{-rt}\, dt\right] = \int_0^\infty x_0\, e^{-(r-\alpha)t}\, dt = x_0/(r-\alpha)\,. \tag{6}$$

Mathematical Background (continued)

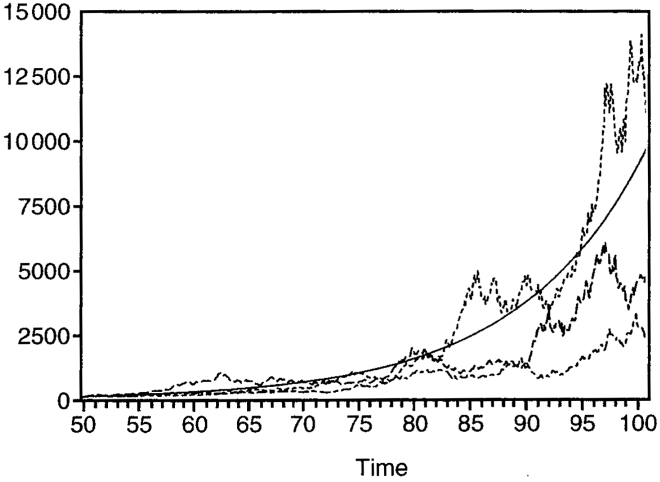
- Fig. 3 shows three sample paths of eq. (5), with $\alpha = .09$ and $\sigma = .2$. (These numbers approximately equal annual expected growth rate and S.D. of the NYSE Index in real terms.) Sample paths were generated by taking a Δt of one month, and using

$$x_t = 1.0075\, x_{t-1} + .0577\, x_{t-1}\, \epsilon_t, \tag{7}$$

with $x_{1950} = 100$. Also shown is the trend line.

- Note that in one of these sample paths the "stock market" outperformed its expected rate of growth, but in the other two it underperformed.

Sample Paths of Geometric Brownian Motion



Mathematical Background (continued)

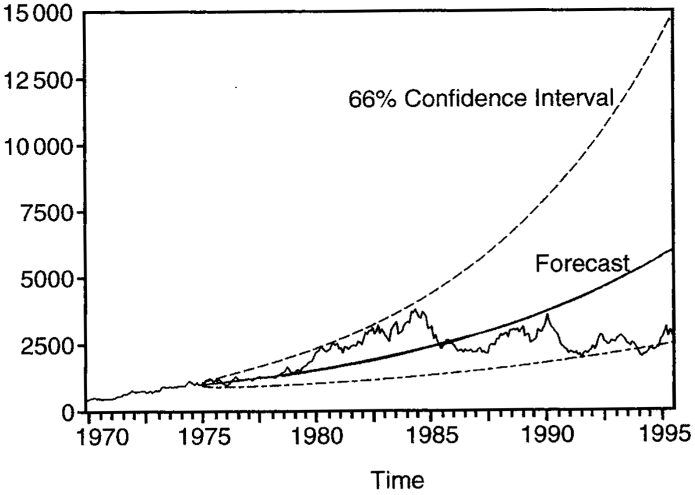
- Figure 4 shows a forecast of this process. The forecasted value of x is given by

$$\hat{x}_{1974+T} = (1.0075)^T x_{1974} \,,$$

where T is in months. Also shown is a 66-percent confidence interval. Since the S.D. of percentage changes in x grows with the square root of the time horizon, bounds of this confidence interval are

$$(1.0075)^T (1.0577)^{\sqrt{T}} x_{1974} \quad \text{and} \quad (1.0075)^T (1.0577)^{-\sqrt{T}} x_{1974} \,.$$

Forecast of Geometric Brownian Motion



Mathematical Background (continued)

- Mean-Reverting Processes: Simplest is

$$dx = \eta\,(\bar{x} - x)\,dt + \sigma\,dz. \tag{8}$$

Here, η is speed of reversion, and $\bar{x}$ is "normal" level of x.

- If $x = x_0$, its expected value at future time t is

$$\mathcal{E}[x_t] = \bar{x} + (x_0 - \bar{x})\,e^{-\eta t}. \tag{9}$$

Also, the variance of $(x_t - \bar{x})$ is

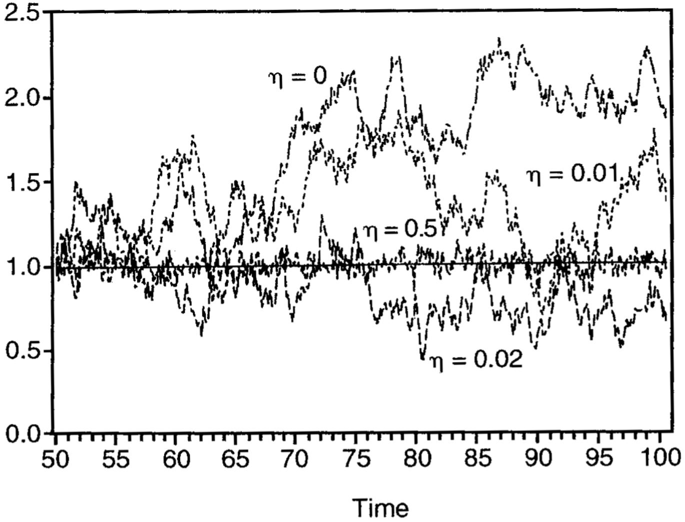
$$\mathcal{V}[x_t - \bar{x}] = \frac{\sigma^2}{2\eta}\,(1 - e^{-2\eta t}). \tag{10}$$

- Note that $\mathcal{E}[x_t] \longrightarrow \bar{x}$ as t becomes large, and variance $\longrightarrow \sigma^2/2\eta$. Also, as $\eta \longrightarrow \infty$, $\mathcal{V}[x_t] \longrightarrow 0$, so x can never deviate from $\bar{x}$. As $\eta \longrightarrow 0$, x becomes a simple Brownian motion, and $\mathcal{V}[x_t] \longrightarrow \sigma^2\,t$.

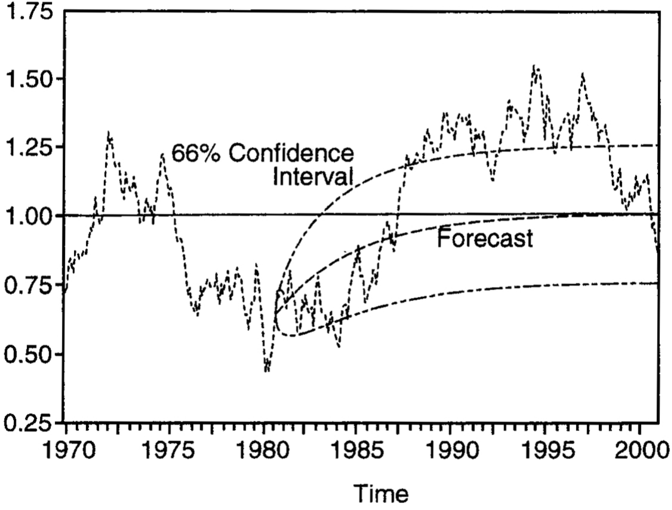
Mathematical Background (continued)

- Fig. 5 shows four sample paths of equation (8) for different values of η. In each case, $\sigma = .05$ in *monthly* terms, $\bar{x} = 1$, and $x(t)$ begins at $x_0 = 1$.
- Fig. 6 shows an optimal forecast for $\eta = .02$, along with a 66-percent confidence interval. Note that after four or five years, the variance of the forecast converges to $\sigma^2/2\eta = .0025/.04 = .065$, so the 66 percent confidence interval ($\pm$ 1 S.D.) converges to the forecast $\pm .25$.

Sample Paths of Mean-Reverting Process $dx = \eta\,(\bar{x} - x)\,dt + \sigma\,dz$



Optimal Forecast of Mean-Reverting Process



Mathematical Background (continued)

- Can generalize eq. (8). For example, one might expect $x(t)$ to revert to $\bar{x}$ but the variance rate to grow with x. Then one could use

$$dx = \eta\,(\bar{x} - x)\,dt + \sigma\,x\,dz. \tag{11}$$

  Or, proportional changes in a variable might be modelled as mean- reverting:

$$dx = \eta\,x\,(\bar{x} - x)\,dt + \sigma\,x\,dz. \tag{12}$$

Mathematical Background (continued)

- Ito's Lemma: Suppose $x(t)$ is an Ito process, i.e.,

$$dx = a(x,t)dt + b(x,t)dz\,, \tag{13}$$

How do we take derivatives of $F(x,t)$?

  - Usual rule of calculus:

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial t}\,dt.$$

  - But suppose we include higher order terms:

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial t}\,dt + \tfrac{1}{2}\frac{\partial^2 F}{\partial x^2}\,(dx)^2 + \tfrac{1}{6}\frac{\partial^3 F}{\partial x^3}\,(dx)^3 + \ldots \tag{14}$$

Mathematical Background (continued)

- In ordinary calculus, these higher order terms all vanish in the limit. Is that the case here? First, substitute eq. (13) for dx to determine $(dx)^2$:

$$(dx)^2 = a^2(x,t)\ (dt)^2 + 2\,a(x,t)\,b(x,t)\ (dt)^{3/2} + b^2(x,t)\ dt.$$

Terms in $(dt)^{3/2}$ and $(dt)^2$ go to zero faster than dt as it becomes infinitesimally small, so we can ignore these terms and write

$$(dx)^2 = b^2(x,t)\,dt.$$

- Every term in the expansion of $(dx)^3$ will include dt to a power greater than 1, and so will go to zero faster than dt in the limit. Likewise for $(dx)^4$, etc. Hence the differential dF is

$$dF = \frac{\partial F}{\partial t}\ dt + \frac{\partial F}{\partial x}\ dx + \tfrac{1}{2}\,\frac{\partial^2 F}{\partial x^2}\ (dx)^2. \qquad (15)$$

Mathematical Background (continued)

- Can write this in expanded form by substituting eq.(13) for dx:

$$dF = \left[\frac{\partial F}{\partial t} + a(x,t)\,\frac{\partial F}{\partial x} + \tfrac{1}{2}\,b^2(x,t)\,\frac{\partial^2 F}{\partial x^2} \right]\; dt$$
$$+\; b(x,t)\,\frac{\partial F}{\partial x}\; dz. \tag{16}$$

- Example: Geometric Brownian Motion. Let's return to the GBM of eq. (5). Use Ito's Lemma to find the process given by $F(x) = \log x$.
- Since $\partial F/\partial t = 0$, $\partial F/\partial x = 1/x$, and $\partial^2 F/\partial x^2 = -1/x^2$, we have from (15):

$$\begin{aligned} dF &= \frac{1}{x}\,dx - \frac{1}{2\,x^2}\,(dx)^2 \\ &= \alpha\,dt + \sigma\,dz - \tfrac{1}{2}\,\sigma^2\,dt = \left(\alpha - \tfrac{1}{2}\,\sigma^2\right)\; dt + \sigma\; dz \end{aligned} \tag{17}$$

Mathematical Background (continued)

- Hence over any interval T, the change in $\log x$ is normally distributed with mean $(\alpha - \frac{1}{2}\sigma^2)\,T$ and variance $\sigma^2\,T$.
- Why is the drift rate of $F(x) = \log x$ less than α? Because $\log x$ is a concave function of x, so with x uncertain, expected value of $\log x$ changes by less than the log of the expected value of x. Uncertainty over x is greater the longer the time horizon, so the expected value of $\log x$ is reduced by an amount that increases with time; hence drift rate is reduced.

Investment Problem

- Invest I, get V, with V a GBM, i.e.,

$$dV = \alpha V dt + \sigma V dz \tag{18}$$

Payout rate is δ and expected rate of return is μ, so $\alpha = \mu - \delta$.

- Want value of the investment opportunity (i.e., the option to invest), $F(V)$, and decision rule V^*.
- Solution by Dynamic Programming: $F(V)$ must satisfy Bellman equation:

$$\rho F \, dt = \mathcal{E}(dF) \,. \tag{19}$$

  - This just says that over interval dt, total expected return on the investment opportunity, $\rho F \, dt$, is equal to expected rate of capital appreciation.
  - Expand dF using Ito's Lemma:

$$dF = F'(V) \, dV + \tfrac{1}{2} F''(V) \, (dV)^2 \,.$$

Investment Problem (continued)

- Substituting eq. (18) for dV and noting that $\mathcal{E}(dz) = 0$ gives

$$\mathcal{E}[dF] = \alpha\, V\, F'(V)\, dt + \tfrac{1}{2}\,\sigma^2\, V^2\, F''(V)\, dt\,.$$

- Hence the Bellman equation becomes (after dividing through by dt):

$$\tfrac{1}{2}\,\sigma^2\, V^2\, F''(V) + \alpha\, V\, F'(V) - \rho\, F = 0\,. \tag{20}$$

- Also, $F(V)$ must satisfy boundary conditions:

$$F(0) = 0 \tag{21}$$

$$F(V^*) = V^* - I \tag{22}$$

$$F'(V^*) = 1 \tag{23}$$

- This is easy to solve, but first, let's do this again using option pricing approach. Then we avoid problem of choosing ρ and α.

Investment Problem (continued)

- Solution by Contingent Claims Analysis.
- Create a risk-free portfolio: Hold option to invest, worth $F(V)$. Short $n = dF/dV$ units of project.

$$\text{Value of portfolio } = \Phi = F - F'(V)V$$

- Short position requires payment of $\delta VF'(V)$ dollars per period. So total return on portfolio is:

$$dF - F'(V)dV - \delta VF'(V)dt$$

Since $dF = F'(V)dV + \frac{1}{2}F''(V)(dV)^2$, total return is:

$$\frac{1}{2}F''(V)(dV)^2 - \delta VF'(V)dt$$

$(dV)^2 = \sigma^2 V^2 dt$, so total return becomes:

$$\frac{1}{2}\sigma^2 V^2 F''(V)dt - \delta VF'(V)dt$$

Investment Problem (continued)

- This return is risk-free. Hence to avoid arbitrage, it must equal $r\Phi dt = r[F - F'(V)V]dt$:

$$\tfrac{1}{2}\sigma^2 V^2 F''(V)dt - \delta V F'(V)dt = r[F - F'(V)V]dt$$

This yields the following differential equation that $F(V)$ must satisfy:

$$\tfrac{1}{2}\sigma^2 V^2 F''(V) + (r - \delta)VF'(V) - rF = 0$$

Boundary conditions:

$$F(0) = 0 \tag{24}$$

$$F(V^*) = V^* - I \tag{25}$$

$$F'(V^*) = 1 \tag{26}$$

Investment Problem (continued)

- Solution. You can check (by substitution) that

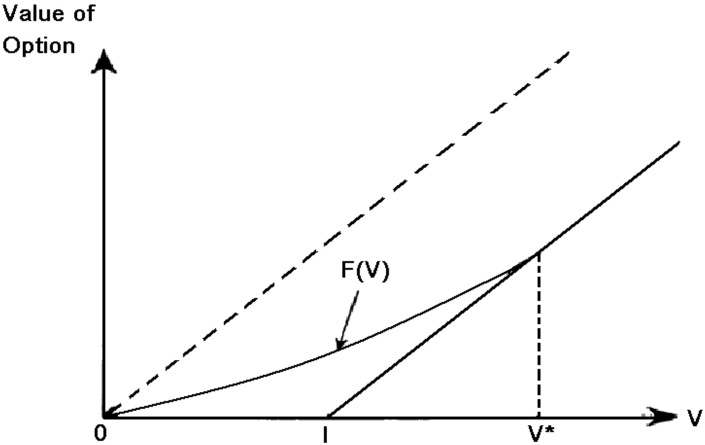
$$
\begin{aligned}
F(V) &= AV^{\beta}, && \text{for } V \le V^* \\
&= V - I, && \text{for } V > V^*
\end{aligned}
$$

$$
\text{where } \beta = \tfrac{1}{2} - \frac{r-\delta}{\sigma^2} + \sqrt{\left(\frac{r-\delta}{\sigma^2} - \tfrac{1}{2}\right)^2 + \frac{2r}{\sigma^2}},
$$

$$
V^* = \frac{\beta I}{\beta - 1}, \qquad A = \frac{V^* - I}{(V^*)^{\beta}}
$$

- Solution is shown in figure.

Graph of $F(V)$



Characteristics of Solution

- Higher σ: Makes F larger, and also makes V^* larger. Now the firm's options to invest (patents, land, etc.) are worth more, but the firm should do less investing. Opportunity cost of investing is higher.

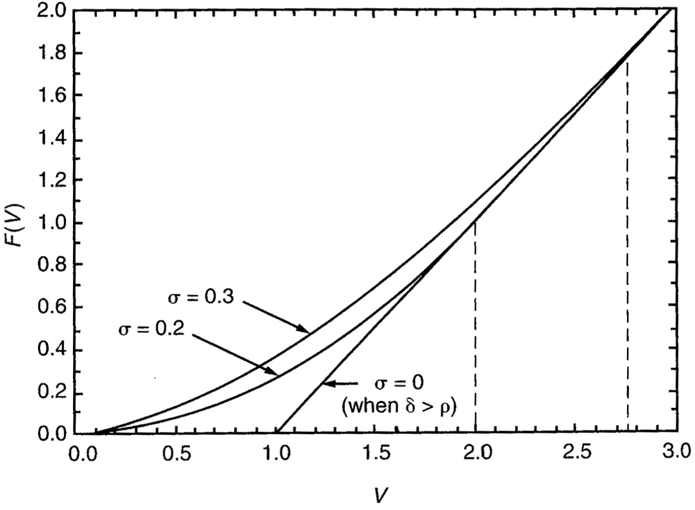
- Higher δ: Higher cash payout rate increases opportunity cost of keeping the option alive. Makes F smaller, V^* smaller. Now the firm's options to invest are worth less, but the firm should do more investing.

- Higher r: Makes F larger and V^* larger. Option value increases because present value of exercise price I decreases (in event of exercise). This increases opportunity cost of "killing" the option, so V^* rises.

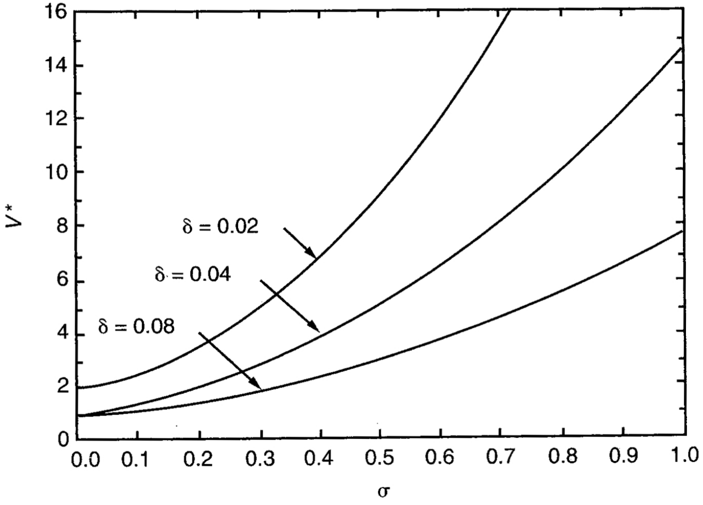
Characteristics of Solution (continued)

- Higher I: Makes F smaller and V^* larger. Note that as $I \to 0$, $F \to V$, and $V^* \to 0$.
- Question: Suppose σ increases. We know this makes V^* larger. What happens to $\mathcal{E}(T^*)$, the expected time until V reaches V^* and investment occurs?
- The following two figures illustrate how $F(V)$ and V^* vary with σ and δ.

Value of Investment Opportunity



Critical Value V^* as a Function of σ



How Important is Option Value?

- Take a factory which costs $1 million ($= I$) to build. Suppose riskless rate is 7% (nominal). Critical value V^* is shown below for different values of σ and δ.

		Payout	**Rate δ**		
		2%	5%	10%	15%
Annual	10%	3.84	1.65	1.13	1.06
Standard Deviation	20%	4.77	2.15	1.40	1.22
of Project Value (σ)	40%	8.06	3.61	2.18	1.73

- For average firm on NYSE, $\sigma = 20\%$.
- Observe that for $\sigma = 20\%$ or 40%, V^* is much larger than I. So how important is option value? Very.

How Important is Option Value? (continued)

- How valuable is option to invest, even if not exercised? Suppose $V = I = \$1$ million. Option value, F, is shown below.

		Payout Rate δ			
		2%	5%	10%	15%
Annual	10%	.46	.18	.05	.02
Standard Deviation	20%	.52	.27	.12	.07
of Project Value (σ)	40%	.65	.44	.28	.20

Mean-Reverting Process

- Suppose V follows the mean-reverting process:

$$dV = \eta\,(\bar{V} - V)\,V\,dt + \sigma\,V\,dz\ , \tag{27}$$

- To find the optimal investment rule, we will use contingent claims analysis.
  - Let $\mu =$ risk-adjusted discount rate for project.
  - Expected rate of growth of V is not constant, but a function of V. Hence the "shortfall," $\delta = \mu - (1/dt)\mathcal{E}(dV)/V$, is a function of V:

$$\delta(V) = \mu - \eta\,(\bar{V} - V)\ . \tag{28}$$

  - The differential equation for $F(V)$ is now

$$\tfrac{1}{2}\sigma^2\,V^2\,F''(V) + [r - \mu + \eta\,(\bar{V} - V)]\,V\,F'(V) - r\,F = 0\ . \tag{29}$$

  - Also, $F(V)$ must satisfy the same boundary conditions (21) – (23) as before, and for the same reasons.

Mean-Reverting Process (continued)

- Solution is a little more complicated. Define a new function $h(V)$ by

$$F(V) = A\,V^{\theta}\,h(V)\,, \tag{30}$$

Substituting this into eq. (29) and rearranging gives the following equation:

$$\begin{aligned} & V^{\theta}\ h(V)\ \left[\tfrac{1}{2}\,\sigma^2\,\theta(\theta-1) + (r-\mu+\eta\,\bar{V})\,\theta - r\right] \\ & + V^{\theta+1}\ \left[\tfrac{1}{2}\,\sigma^2\,V\,h''(V) + (\sigma^2\theta + r - \mu + \eta\,\bar{V} - \eta\,V)\,h'(V)\right. \\ & \qquad \left. -\ \eta\,\theta\,h(V)\right] = 0\,. \end{aligned} \tag{31}$$

Mean-Reverting Process (continued)

- Eq. (31) must hold for any value of V, so the bracketed terms in both the first and second lines must equal zero. First choose θ to set the bracketed terms in the first line equal to zero:

$$\tfrac{1}{2}\,\sigma^2\,\theta(\theta-1)+(r-\mu+\eta\,\bar{V})\,\theta-r=0\,.$$

To satisfy the boundary condition that $F(0)=0$, we use the positive solution:

$$\theta=\tfrac{1}{2}+(\mu-r-\eta\bar{V})/\sigma^2+\sqrt{\left[(r-\mu+\eta\bar{V})/\sigma^2-\tfrac{1}{2}\right]^2+2r/\sigma^2} \tag{32}$$

- From the second line of eq. (31),

$$\tfrac{1}{2}\,\sigma^2\,V\,h''(V)+(\sigma^2\,\theta+r-\mu+\eta\,\bar{V}-\eta\,V)\,h'(V)-\eta\,\theta\,h(V)=0 \tag{33}$$

Mean-Reverting Process (continued)

Make the substitution $x = 2\eta V/\sigma^2$, to transform eq.(33) into a standard form. Let $h(V) = g(x)$, so that $h'(V) = (2\eta/\sigma^2)\, g'(x)$ and $h''(V) = (2\eta/\sigma^2)^2\, g''(x)$. Then (33) becomes

$$x\, g''(x) + (b - x)\, g'(x) - \theta\, g(x) = 0 \ , \tag{34}$$

where

$$b = 2\theta + 2\,(r - \mu + \eta \bar{V})/\sigma^2 \ .$$

- Eq. (34) is Kummer's Equation. Its solution is the confluent hypergeometric function $H(x; \theta, b(\theta))$, which has the following series representation:

$$H(x; \theta, b) = 1 + \frac{\theta}{b} x + \frac{\theta(\theta+1)}{b(b+1)} \frac{x^2}{2!} + \frac{\theta(\theta+1)(\theta+2)}{b(b+1)(b+2)} \frac{x^3}{3!} + \ldots \ . \tag{35}$$

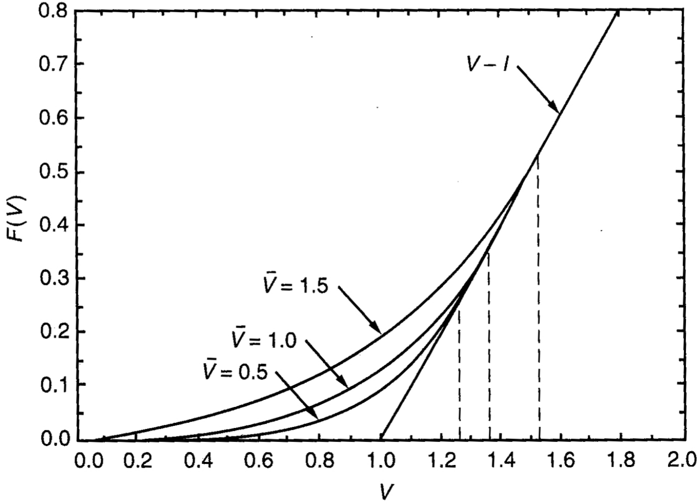
Mean-Reverting Process (continued)

- We have verified that the solution to equation (29) is indeed of the form of equation (30). Solution is

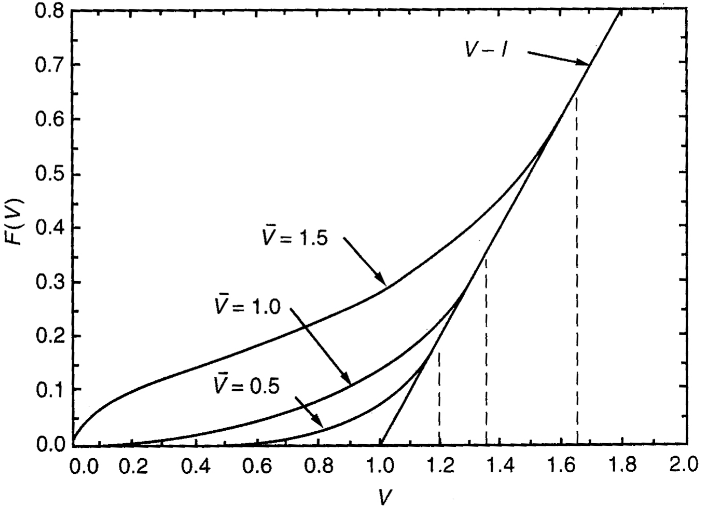
$$F(V) = A\,V^{\theta}\,H\left(\frac{2\eta}{\sigma^2}V; \theta, b\right) , \tag{36}$$

  where A is a constant yet to be determined.
- We can find A, as well as the critical value V^*, from the remaining two boundary conditions, that is, $F(V^*) = V^* - I$ and $F_V(V^*) = 1$. Because the confluent hypergeometric function is an infinite series, A and V^* must be found numerically.
- Look at several numerical solutions. Set $I = 1$, $r = .04$, $\mu = .08$, and $\sigma = .2$. We will vary η and $\bar{V}$.

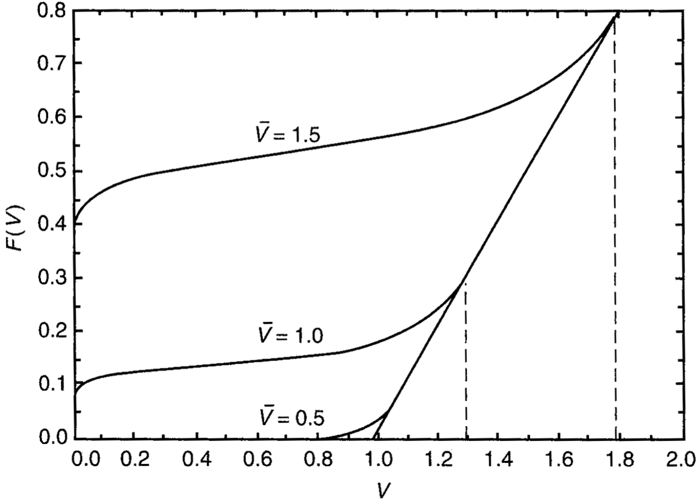
Mean Reversion— $F(V)$ for $\eta = 0.05$ and $\bar{V} = 0.5$, 1.0, and 1.5



Mean Reversion— $F(V)$ for $\eta = 0.1$ and $\bar{V} = 0.5$, 1.0, and 1.5



Mean Reversion— $F(V)$ for $\eta = 0.5$ and $\bar{V} = 0.5$, 1.0, and 1.5



Crtical Value V^* as a Function of η for $\mu = 0.08$ and $\bar{V} = 0.1$, 1.0, and 1.5